Prime Graphs of Finite Groups

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Prime graphs of A_5 and $\mathrm{PSL}_2(8)$

Figure : A disconnected graph with three connected components

The prime graph of A_8

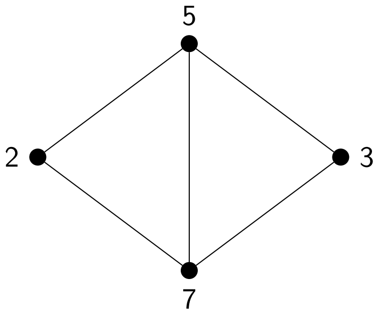

Figure : Diamond

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- (Lewis, Meng, 2012) If G is solvable and $\Delta(G) \cong C_4$ is a square, then $G = A \times B$ where $\rho(A) = \{p, q\}$ and $\rho(B) = \{r, s\}$.

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   - *(ii)* $G \cong H \times K$, where $H \in \{\mathrm{A}_5, \mathrm{PSL}_2(8)\}$, K is solvable, $\Delta(K)$ is disconnected with two connected components, $|\rho(K)| = 2$ and $\rho(H) \cap \rho(K) = \emptyset$.

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The first prime graph with 5 vertices

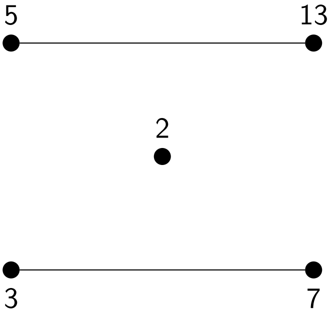

Figure : The prime graph of $\mathrm{PSL}_2(2^6)$

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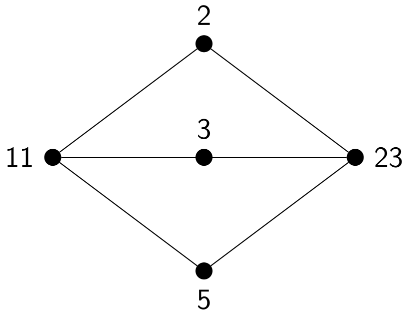

Figure : The prime graph of $\mathrm{A}_5 \times 23^{1+2} : 11$

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Let G be an almost simple group with simple socle S. Suppose that $\Delta(G)$ has no triangle. Then $S \cong \mathrm{PSL}_2(q)$ with $q \geq 4$ a prime power, $\pi(G) = \pi(S)$ and $|\pi(G)| \leq 5$.

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Lemma 3 (Reduction)

Let G be a nonsolvable group and let N be the solvable radical of G. Suppose that $\Delta(G)$ has no triangle. Then there exists a normal subgroup M of G such that $M/N \cong \mathrm{PSL}_2(q)$, with $q \geq 4$ a prime power, G/N is an almost simple group with socle M/N, and $\rho(G) = \rho(M)$.

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Lemma 4 (Key Lemma)

Let G be a nonsolvable group and let N be a normal subgroup of G such that G/N is nonabelian simple. Let $\theta \in \mathrm{Irr}(N)$. Then either $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(G/N)$ for some $\chi \in \mathrm{Irr}(G|\theta)$ or θ extends to G and $G/N \cong \mathrm{A}_5$ or $\mathrm{PSL}_2(8)$.

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Possible extensions and related questions

- Classify prime graphs of finite groups which is K_n-free for some $n \geq 4$.
- (Conjecture) If $\Delta(G)$ is K_n-free with $n \geq 4$, then $|\rho(G)| \leq 2n - 1$.
- (Strengthened Huppert's $\rho - \sigma$ Conjecture) $\rho(G) \leq 2\sigma(G) + 1$.
- Classify prime graphs of finite groups which contain a small number $t(G)$ of triangles.
- (Conjecture) For any group G, we have $|\rho(G)| \leq t(G) + 5$.

Thank you

Thank you for your attention!